

Quadratic-quartic functional equations in RN-spaces

M. Bavand Savadkouhi

Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
e-mail: bavand.m@gmail.com

M. Eshaghi Gordji

Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
e-mail: madjid.eshaghi@gmail.com

Choonkil Park

Department of Mathematics, Hanyang University,
Seoul 133-791, South Korea
e-mail: baak@hanyang.ac.kr

Abstract. In this paper, we obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t -norms

$$f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 2[f(2x) - 4f(x)] - 6f(y).$$

1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam [33] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *, d)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x.y), h(x)*h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [15] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ and some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is \mathbb{R} -linear. In 1978, Th. M. Rassias [27] provided a generalization of the Hyers' theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [10] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as

⁰2000 Mathematics Subject Classification: 46S40, 39B72, 54E40.

⁰Keywords: Quadratic-quartic functional equation; Random normed space; Stability.

Hyers-Ulam-Rassias stability of functional equations (see [1, 2, 3, 11, 16, 17, 28, 29]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \quad (1.1)$$

is related to a symmetric bi-additive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [1, 18]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \quad (1.2)$$

The Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \rightarrow B$, where A is a normed space and B is a Banach space (see [32]). Cholewa [5] noticed that the theorem of Skof is still true if relevant domain A is replaced an abelian group. In [7], Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.1). Grabiec [12] has generalized these results mentioned above. In [26], W. Park and J. Bae considered the following quartic functional equation

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y) + 6f(y)] - 6f(x). \quad (1.3)$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive mapping $M : X^4 \rightarrow Y$ such that $f(x) = M(x, x, x, x)$ for all x . It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation (see also [6]). In addition, Kim [19] has obtained the Hyers-Ulam-Rassias stability for a mixed type of quartic and quadratic functional equation.

The Hyers-Ulam-Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [20]-[25]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M .

The aim of this paper is to investigate the stability of the additive-quadratic functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary continuous t -norms.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [4, 21, 22, 30, 31]. Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1]$ such that F is left-continuous and non-decreasing on \mathbb{R} , $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x , that is, $l^-f(x) = \lim_{t \rightarrow x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in \mathbb{R} . The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([30]). A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a continuous t -norm) if T satisfies the following conditions:

- (a) T is commutative and associative;
- (b) T is continuous;
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t -norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t -norm). Recall (see [13, 14]) that if T is a t -norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i)$ for $n \geq 2$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known ([14]) that for the Lukasiewicz t -norm the following implication holds:

$$\lim_{n \rightarrow \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^\infty (1 - x_n) < \infty.$$

Definition 1.2. ([31]). A random normed space (briefly, RN-space) is a triple (X, μ, T) , where X is a vector space, T is a continuous t -norm and μ is a mapping from X into D^+ such that the following conditions hold:

- (RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
- (RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M) , where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t -norm. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be an RN-space.

- (1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
- (2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
- (3) An RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4. ([30]). If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Recently, M. Eshaghi Gordji et al. establish the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [8] and [9]).

In this paper, we deal with the following functional equation

$$f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 2[f(2x) - 4f(x)] - 6f(y) \quad (1.4)$$

on RN-spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of (1.4).

In Section 2, we investigate the general solution of the functional equation (1.4) when f is a mapping between vector spaces and in Section 3, we establish the stability of the functional equation (1.4) in RN-spaces.

2. GENERAL SOLUTION

We need the following lemma for solution of (1.4). Throughout this section X and Y are vector spaces.

Lemma 2.1. If a mapping $f : X \rightarrow Y$ satisfies (1.4) for all $x, y \in X$, then f is quadratic-quartic.

Proof. We show that the mappings $g : X \longrightarrow Y$ defined by $g(x) := f(2x) - 16f(x)$ and $h : X \longrightarrow Y$ defined by $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.4), we have $f(0) = 0$. Putting $x = 0$ in (1.4), we get $f(-y) = f(y)$. Thus the mapping f is even. Replacing y by $2y$ in (1.4), we get

$$f(2x + 2y) + f(2x - 2y) = 4[f(x + 2y) + f(x - 2y)] + 2[f(2x) - 4f(x)] - 6f(2y) \quad (2.1)$$

for all $x, y \in X$. Interchanging x with y in (1.4), we obtain

$$f(2y + x) + f(2y - x) = 4[f(y + x) + f(y - x)] + 2[f(2y) - 4f(y)] - 6f(x) \quad (2.2)$$

for all $x, y \in X$. Since f is even, by (2.2), one gets

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 2[f(2y) - 4f(y)] - 6f(x) \quad (2.3)$$

for all $x, y \in X$. It follows from (2.1) and (2.3) that

$$[f(2(x + y)) - 16f(x + y)] + [f(2(x - y)) - 16f(x - y)] = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)]$$

for all $x, y \in X$. This means that

$$g(x + y) + g(x - y) = 2g(x) + 2g(y)$$

for all $x, y \in X$. Therefore, the mapping $g : X \rightarrow Y$ is quadratic.

To prove that $h : X \rightarrow Y$ is quartic, we have to show that

$$h(x + 2y) + h(x - 2y) = 4[h(x + y) + h(x - y) + 6h(y)] - 6h(x)$$

for all $x, y \in X$. Since f is even, the mapping h is even. Now if we interchange x with y in the last equation, we get

$$h(2x + y) + h(2x - y) = 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y) \quad (2.4)$$

for all $x, y \in X$. Thus it is enough to prove that h satisfies in (2.4). Replacing x and y by $2x$ and $2y$ in (1.4), respectively, we obtain

$$f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 2[f(4x) - 4f(2x)] - 6f(2y) \quad (2.5)$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$,

$$f(4x) = 20f(2x) - 64f(x) \quad (2.6)$$

for all $x \in X$. By (2.5) and (2.6), we get

$$f(2(2x + y)) + f(2(2x - y)) = 4[f(2(x + y)) + f(2(x - y))] + 32[f(2x) - 4f(x)] - 6f(2y) \quad (2.7)$$

for all $x, y \in X$. By multiplying both sides of (1.4) by 4, we get

$$4[f(2x + y) + f(2x - y)] = 16[f(x + y) + f(x - y)] + 8[f(2x) - 4f(x)] - 24f(y)$$

for all $x, y \in X$. If we subtract the last equation from (2.7), we obtain

$$\begin{aligned} h(2x + y) + h(2x - y) &= [f(2(2x + y)) - 4f(2x + y)] + [f(2(2x - y)) - 4f(2x - y)] \\ &= 4[f(2(x + y)) - 4f(x + y)] + 4[f(2(x - y)) - 4f(x - y)] \\ &\quad + 24[f(2x) - 4f(x)] - 6[f(2y) - 4f(y)] \\ &= 4[h(x + y) + h(x - y) + 6h(x)] - 6h(y) \end{aligned}$$

for all $x, y \in X$.

Therefore, the mapping $h : X \rightarrow Y$ is quartic. This completes the proof of the lemma. \square

Theorem 2.2. *A mapping $f : X \rightarrow Y$ satisfies (1.4) for all $x, y \in X$ if and only if there exist a unique symmetric multi-additive mapping $M : X^4 \rightarrow Y$ and a unique symmetric bi-additive mapping $B : X \times X \rightarrow Y$ such that*

$$f(x) = M(x, x, x, x) + B(x, x)$$

for all $x \in X$.

Proof. Let f satisfies (1.4) and assume that $g, h : X \rightarrow Y$ are mappings defined by

$$g(x) := f(2x) - 16f(x), \quad h(x) := f(2x) - 4f(x)$$

for all $x \in X$. By Lemma 2.1, we obtain that the mappings g and h are quadratic and quartic, respectively, and

$$f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x)$$

for all $x \in X$.

Therefore, there exist a unique symmetric multi-additive mapping $M : X^4 \rightarrow Y$ and a unique symmetric bi-additive mapping $B : X \times X \rightarrow Y$ such that $\frac{1}{12}h(x) = M(x, x, x, x)$ and $\frac{-1}{12}g(x) = B(x, x)$ for all $x \in X$ (see [1, 26]). So

$$f(x) = M(x, x, x, x) + B(x, x)$$

for all $x \in X$. The proof of the converse is obvious. \square

3. STABILITY

Throughout this section, assume that X is a real linear space and (Y, μ, T) is a complete RN-space.

Theorem 3.1. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x, y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x, y}(t) \quad (3.1)$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} \lim_{n \rightarrow \infty} T_{i=1}^{\infty} (\rho_{2^{n+i-1}x, 2^{n+i-1}x}(\frac{2^{2n+i}t}{4}) + \rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}(2^{2n+i}t) \\ + \rho_{0, 2^{n+i-1}x}(\frac{3 \cdot 2^{2n+i}t}{4})) = 1 \end{aligned} \quad (3.2)$$

and

$$\lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{2n}t) = 1 \quad (3.3)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1(x)}(t) \geq T_{i=1}^{\infty} (\rho_{2^{i-1}x, 2^{i-1}x}(\frac{2^i t}{4}) + \rho_{2^{i-1}x, 2 \cdot 2^{i-1}x}(2^i t) + \rho_{0, 2^{i-1}x}(\frac{3 \cdot 2^i t}{4})) \quad (3.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.1), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x, x}(t) \quad (3.5)$$

for all $x \in X$. Letting $y = 2x$ in (3.1), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x, 2x}(t) \quad (3.6)$$

for all $x \in X$. Putting $x = 0$ in (3.1), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \quad (3.7)$$

for all $y \in X$. Replacing y by x in (3.7), we see that

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \quad (3.8)$$

for all $x \in X$. It follows from (3.6) and (3.8) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right) \quad (3.9)$$

for all $x \in X$. If we add (3.5) to (3.9), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \rho_{x,x}\left(\frac{t}{4}\right) + \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right). \quad (3.10)$$

Let

$$\psi_{x,x}(t) = \rho_{x,x}\left(\frac{t}{4}\right) + \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right) \quad (3.11)$$

for all $x \in X$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t) \quad (3.12)$$

for all $x \in X$ and all $t > 0$. Let $g : X \rightarrow Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$\mu_{g(2x)-4g(x)}(t) \geq \psi_{x,x}(t) \quad (3.13)$$

for all $x \in X$. Thus we have

$$\mu_{\frac{g(2x)}{2^2}-g(x)}(t) \geq \psi_{x,x}(2^2t) \quad (3.14)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{\frac{g(2^{k+1}x)}{2^{2(k+1)}}-\frac{g(2^kx)}{2^{2k}}}(t) \geq \psi_{2^kx,2^kx}(2^{2(k+1)}t) \quad (3.15)$$

for all $x \in X$ and all $k \in \mathbb{N}$. This means that

$$\mu_{\frac{g(2^{k+1}x)}{2^{2(k+1)}}-\frac{g(2^kx)}{2^{2k}}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^kx,2^kx}(2^{k+1}t) \quad (3.16)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$, it follows

$$\begin{aligned} \mu_{\frac{g(2^n x)}{2^{2n}}-g(x)}(t) &\geq T_{k=0}^{n-1} \left(\mu_{\frac{g(2^{k+1}x)}{2^{2(k+1)}}-\frac{g(2^kx)}{2^{2k}}}\left(\frac{t}{2^{k+1}}\right) \right) \geq T_{k=0}^{n-1} (\psi_{2^kx,2^kx}(2^{k+1}t)) \\ &= T_{i=1}^n (\psi_{2^{i-1}x,2^{i-1}x}(2^i t)) \end{aligned} \quad (3.17)$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{g(2^n x)}{2^{2n}}\}$, we replace x with $2^m x$ in (3.17) to obtain that

$$\mu_{\frac{g(2^{n+m}x)}{2^{2(n+m)}}-\frac{g(2^m x)}{2^{2m}}}(t) \geq T_{i=1}^n (\psi_{2^{i+m-1}x,2^{i+m-1}x}(2^{i+2m}t)). \quad (3.18)$$

Since the right hand side of the inequality (3.18) tends to 1 as m and n tend to infinity, the sequence $\{\frac{g(2^n x)}{2^{2n}}\}$ is a Cauchy sequence. Thus we may define $Q_1(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^{2n}}$ for all $x \in X$. Now we show that Q_1 is a quadratic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.1), *respectively*, we get

$$\mu_{g(2x+y)+g(2x-y)-4g(x+y)-4g(x-y)-2g(2x)+8g(x)+6g(y)}(t) \geq \rho_{2^n x, 2^n y}(2^{2n}t). \quad (3.19)$$

Taking the limit as $n \rightarrow \infty$, we find that Q_1 satisfies (1.4) for all $x, y \in X$. By Lemma 2.1, the mapping $Q_1 : X \rightarrow Y$ is quadratic.

Letting the limit as $n \rightarrow \infty$ in (3.17), we get (3.4) by (3.11).

Finally, to prove the uniqueness of the quadratic mapping Q_1 subject to (3.4), let us assume that there exists another quadratic mapping Q'_1 which satisfies (3.4). Since $Q_1(2^n x) = 2^{2n} Q_1(x)$, $Q'_1(2^n x) = 2^{2n} Q'_1(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (3.4), it follows that

$$\begin{aligned} \mu_{Q_1(x)-Q'_1(x)}(2t) &= \mu_{Q_1(2^n x)-Q'_1(2^n x)}(2^{2n+1}t) \\ &\geq T(\mu_{Q_1(2^n x)-g(2^n x)}(2^{2n}t), \mu_{g(2^n x)-Q'_1(2^n x)}(2^{2n}t)) \\ &\geq T(T_{i=1}^\infty(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(\frac{2^{2n+i}t}{4}) + \rho_{2^{i+n-1}x, 2 \cdot 2^{i+n-1}x}(2^{2n+i}t) \\ &\quad + \rho_{0, 2^{i+n-1}x}(\frac{3 \cdot 2^{2n+i}t}{4})), T_{i=1}^\infty(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(\frac{2^{2n+i}t}{4}) \\ &\quad + \rho_{2^{i+n-1}x, 2 \cdot 2^{i+n-1}x}(2^{2n+i}t) + \rho_{0, 2^{i+n-1}x}(\frac{3 \cdot 2^{2n+i}t}{4}))) \end{aligned} \quad (3.20)$$

for all $x \in X$ and all $t > 0$. By letting $n \rightarrow \infty$ in (3.20), we conclude that $Q_1 = Q'_1$. \square

Theorem 3.2. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x, y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x, y}(t) \quad (3.21)$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} \lim_{n \rightarrow \infty} T_{i=1}^\infty(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(\frac{2^{4n+3i}t}{4}) + \rho_{2^{n+i-1}x, 2 \cdot 2^{n+i-1}x}(2^{4n+3i}t) \\ + \rho_{0, 2^{n+i-1}x}(\frac{3 \cdot 2^{4n+3i}t}{4})) = 1 \end{aligned} \quad (3.22)$$

and

$$\lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{4n}t) = 1 \quad (3.23)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q_2 : X \rightarrow Y$ such that

$$\mu_{f(2x)-4f(x)-Q_2(x)}(t) \geq T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(\frac{2^{3i}t}{4}) + \rho_{2^{i-1}x, 2 \cdot 2^{i-1}x}(2^{3i}t) + \rho_{0, 2^{i-1}x}(\frac{3 \cdot 2^{3i}t}{4})) \quad (3.24)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.21), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x, x}(t) \quad (3.25)$$

for all $x \in X$. Letting $y = 2x$ in (3.21), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x, 2x}(t) \quad (3.26)$$

for all $x \in X$. Putting $x = 0$ in (3.21), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0, y}(t) \quad (3.27)$$

for all $y \in X$. Replacing y by x in (3.27), we get

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0, x}(t) \quad (3.28)$$

for all $x \in X$. It follows from (3.6) and (3.28) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq \rho_{x, 2x}(t) + \rho_{0, x}(\frac{3t}{4}) \quad (3.29)$$

for all $x \in X$. If we add (3.25) to (3.29), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \rho_{x,x}\left(\frac{t}{4}\right) + \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right). \quad (3.30)$$

Let

$$\psi_{x,x}(t) = \rho_{x,x}\left(\frac{t}{4}\right) + \rho_{x,2x}(t) + \rho_{0,x}\left(\frac{3t}{4}\right) \quad (3.31)$$

for all $x \in X$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t) \quad (3.32)$$

for all $x \in X$ and all $t > 0$. Let $h : X \rightarrow Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$. Then we conclude that

$$\mu_{h(2x)-16h(x)}(t) \geq \psi_{x,x}(t) \quad (3.33)$$

for all $x \in X$. Thus we have

$$\mu_{\frac{h(2x)}{2^4}-h(x)}(t) \geq \psi_{x,x}(2^4t) \quad (3.34)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{\frac{h(2^{k+1}x)}{2^{4(k+1)}}-\frac{h(2^kx)}{2^{4k}}}(t) \geq \psi_{2^kx,2^kx}(2^{4(k+1)}t) \quad (3.35)$$

for all $x \in X$ and all $k \in \mathbb{N}$. This means that

$$\mu_{\frac{h(2^{k+1}x)}{2^{4(k+1)}}-\frac{h(2^kx)}{2^{4k}}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^kx,2^kx}(2^{3(k+1)}t) \quad (3.36)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$, it follows

$$\begin{aligned} \mu_{\frac{h(2^n x)}{2^{4n}}-h(x)}(t) &\geq T_{k=0}^{n-1}\left(\mu_{\frac{h(2^{k+1}x)}{2^{4(k+1)}}-\frac{h(2^kx)}{2^{4k}}}\left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=0}^{n-1}(\psi_{2^kx,2^kx}(2^{3(k+1)}t)) \\ &= T_{i=1}^n(\psi_{2^{i-1}x,2^{i-1}x}(2^{3i}t)) \end{aligned} \quad (3.37)$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{h(2^n x)}{2^{4n}}\}$, we replace x with $2^m x$ in (3.37) to obtain that

$$\mu_{\frac{h(2^{n+m}x)}{2^{4(n+m)}}-\frac{h(2^m x)}{2^{4m}}}(t) \geq T_{i=1}^n(\psi_{2^{i+m-1}x,2^{i+m-1}x}(2^{3i+4m}t)). \quad (3.38)$$

Since the right hand side of the inequality (3.38) tends to 1 as m and n tend to infinity, the sequence $\{\frac{h(2^n x)}{2^{4n}}\}$ is a Cauchy sequence. Thus we may define $Q_2(x) = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{2^{4n}}$ for all $x \in X$. Now we show that Q_2 is a quartic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.21), *respectively*, we get

$$\mu_{h(2x+y)+h(2x-y)-4h(x+y)-4h(x-y)-2h(2x)+8h(x)+6h(y)}(t) \geq \rho_{2^n x, 2^n y}(2^{4n}t). \quad (3.39)$$

Taking the limit as $n \rightarrow \infty$, we find that Q_2 satisfies (1.4) for all $x, y \in X$. By Lemma 2.1 we get that the mapping $Q_2 : X \rightarrow Y$ is quartic.

Letting the limit as $n \rightarrow \infty$ in (3.37), we get (3.24) by (3.31).

Finally, to prove the uniqueness of the quartic mapping Q_2 subject to (3.24), let us assume that there exists a quartic mapping Q'_2 which satisfies (3.24). Since $Q_2(2^n x) = 2^{4n} Q_2(x)$ and

$Q'_2(2^n x) = 2^{4n} Q'_2(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (3.24), it follows that

$$\begin{aligned} \mu_{Q_2(x)-Q'_2(x)}(2t) &= \mu_{Q_2(2^n x)-Q'_2(2^n x)}(2^{4n+1}t) \\ &\geq T(\mu_{Q_2(2^n x)-h(2^n x)}(2^{4n}t), \mu_{h(2^n x)-Q'_2(2^n x)}(2^{4n}t)) \\ &\geq T(T_{i=1}^\infty(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(\frac{2^{4n+3i}t}{4}) + \rho_{2^{i+n-1}x, 2.2^{i+n-1}x}(2^{4n+3i}t) \\ &\quad + \rho_{0, 2^{i+n-1}x}(\frac{3.2^{4n+3i}t}{4})), T_{i=1}^\infty(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(\frac{2^{4n+3i}t}{4}) \\ &\quad + \rho_{2^{i+n-1}x, 2.2^{i+n-1}x}(2^{4n+3i}t) + \rho_{0, 2^{i+n-1}x}(\frac{3.2^{4n+3i}t}{4}))) \end{aligned} \quad (3.40)$$

for all $x \in X$ and all $t > 0$. By letting $n \rightarrow \infty$ in (3.40), we get that $Q_2 = Q'_2$. \square

Theorem 3.3. *Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x, y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x, y}(t) \quad (3.41)$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned} \lim_{n \rightarrow \infty} T_{i=1}^\infty(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(\frac{2^{4n+3i}t}{4}) + \rho_{2^{n+i-1}x, 2.2^{n+i-1}x}(2^{4n+3i}t) + \rho_{0, 2^{n+i-1}x}(\frac{3.2^{4n+3i}t}{4})) \\ = 1 \\ = \lim_{n \rightarrow \infty} T_{i=1}^\infty(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(\frac{2^{2n+i}t}{4}) + \rho_{2^{n+i-1}x, 2.2^{n+i-1}x}(2^{2n+i}t) + \rho_{0, 2^{n+i-1}x}(\frac{3.2^{2n+i}t}{4})) \end{aligned} \quad (3.42)$$

and

$$\lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{4n}t) = 1 = \lim_{n \rightarrow \infty} \rho_{2^n x, 2^n y}(2^{2n}t) \quad (3.43)$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $Q_1 : X \rightarrow Y$ and a unique quartic mapping $Q_2 : X \rightarrow Y$ such that

$$\begin{aligned} \mu_{f(x)-Q_1(x)-Q_2(x)}(t) \\ \geq T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(3.2^i t) + \rho_{2^{i-1}x, 2.2^{i-1}x}(12.2^i t) + \rho_{0, 2^{i-1}x}(9.2^i t)) \\ + T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(3.2^{3i}) + \rho_{2^{i-1}x, 2.2^{i-1}x}(12.2^{3i} t) + \rho_{0, 2^{i-1}x}(9.2^{3i})) \end{aligned} \quad (3.44)$$

for all $x \in X$ and all $t > 0$.

Proof. By Theorems 3.1 and 3.2, there exist a quadratic mapping $Q'_1 : X \rightarrow Y$ and a quartic mapping $Q'_2 : X \rightarrow Y$ such that

$$\mu_{f(2x)-16f(x)-Q'_1(x)}(t) \geq T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(\frac{2^i t}{4}) + \rho_{2^{i-1}x, 2.2^{i-1}x}(2^i t) + \rho_{0, 2^{i-1}x}(\frac{3.2^i t}{4}))$$

and

$$\mu_{f(2x)-4f(x)-Q'_2(x)}(t) \geq T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(\frac{2^{3i} t}{4}) + \rho_{2^{i-1}x, 2.2^{i-1}x}(2^{3i} t) + \rho_{0, 2^{i-1}x}(\frac{3.2^{3i} t}{4}))$$

for all $x \in X$ and all $t > 0$. So it follows from the last inequalities that

$$\begin{aligned} \mu_{f(x)+\frac{1}{12}Q'_1(x)-\frac{1}{12}Q'_2(x)}(t) \\ \geq T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(3.2^i t) + \rho_{2^{i-1}x, 2.2^{i-1}x}(12.2^i t) + \rho_{0, 2^{i-1}x}(9.2^i t)) \\ + T_{i=1}^\infty(\rho_{2^{i-1}x, 2^{i-1}x}(3.2^{3i}) + \rho_{2^{i-1}x, 2.2^{i-1}x}(12.2^{3i} t) + \rho_{0, 2^{i-1}x}(9.2^{3i})) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we obtain (3.46) by letting $Q_1(x) = -\frac{1}{12}Q'_1(x)$ and $Q_2(x) = \frac{1}{12}Q'_2(x)$ for all $x \in X$. The uniqueness property of Q_1 and Q_2 , are trivial. \square

4. ACKNOWLEDGEMENT

The third author was supported by Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00041).

REFERENCES

- [1] J. Aczel and J. Dhombres, Functional Equations in Several Variables, *Cambridge Univ. Press*, 1989.
- [2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan.* **2** (1950), 64–66.
- [3] D. G. Bourgin, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.* **57** (1951), 223–237.
- [4] S. S. Chang, Y. J. Cho and S. M. Kang, Nonlinear Operator Theory in Probabilistic Metric Spaces, *Nova Science Publishers Inc. New York*, 2001.
- [5] P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76–86.
- [6] J. K. Chung and P. K. Sahoo, On the general solution of a quartic functional equation, *Bull. Korean Math. Soc.* **40** (2003), 565–576.
- [7] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59–64.
- [8] M. Eshaghi Gordji, J. M. Rassias and M. Bavand Savadkouhi, Stability of a mixed type additive and quadratic functional equation in random normed spaces (preprint).
- [9] M. Eshaghi Gordji, J. M. Rassias and M. Bavand Savadkouhi, Approximation of the quadratic and cubic functional equation in RN-spaces (preprint).
- [10] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (1991), 431–434.
- [11] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431–436.
- [12] A. Grabiec, The generalized Hyers-Ulam stability of a class of functional equations, *Publ. Math. Debrecen* **48** (1996), 217–235.
- [13] O. Hadžić and E. Pap, Fixed Point Theory in PM Spaces, *Kluwer Academic Publishers, Dordrecht*, 2001.
- [14] O. Hadžić, E. Pap and M. Budincević, Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, *Kybernetika*, **38** (3) (2002), 363–381.
- [15] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. USA* **27** (1941), 222–224.
- [16] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser, Basel*, 1998.
- [17] G. Isac and Th. M. Rassias, On the Hyers-Ulam stability of ψ -additive mappings, *J. Approx. Theory* **72** (1993), 131–137.
- [18] Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.* **27** (1995), 368–372.
- [19] H. Kim, On the stability problem for a mixed type of quartic and quadratic functional equation, *J. Math. Anal. Appl.* **324** (2006), 358–372.

- [20] D. Miheţ, The probabilistic stability for a functional equation in a single variable, *Acta Math. Hungar.* (in press).
- [21] D. Miheţ, The fixed point method for fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems* (in press).
- [22] D. Miheţ and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.* **343** (2008), 567–572.
- [23] M. Mirmostafae, M. Mirzavaziri and M. S. Moslehian, Fuzzy stability of the Jensen functional equation *Fuzzy Sets and Systems*, **159** (2008), 730–738.
- [24] A. K. Mirmostafae and M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems* **159** (2008), 720–729.
- [25] A. K. Mirmostafae, M. S. Moslehian, Fuzzy approximately cubic mappings, *Inform. Sci.* **178** (2008), 3791–3798.
- [26] W. Park, J. Bae, On a bi-quadratic functional equation and its stability, *Nonlinear Anal. –TMA* **62** (2005), 643–654.
- [27] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
- [28] Th. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Math. Appl.* **62** (2000), 23–130.
- [29] Th. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.* **251** (2000), 264–284.
- [30] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, *Elsevier, North Holland, New York*, 1983.
- [31] A. N. Sherstnev, On the notion of a random normed space, *Dokl. Akad. Nauk SSSR* **149** (1963), 280–283 (in Russian).
- [32] F. Skof, Propriet  locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano*, **53** (1983), 113–129.
- [33] S. M. Ulam, Problems in Modern Mathematics, *Chapter VI, Science ed., Wiley, New York*, 1940.